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# SLOW DISSIPATIVE EVOLUTION OF THE MOTION OF A VISCOELASTIC SPHERE IN THE RESTRICTED CIRCULAR THREE-BODY PROBLEM†

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The restricted circular three-body problem is considered. In this problem two massive bodies, simulated by point masses with masses of unity and $\mu$, move in specified circular orbits around a common centre of mass, while a third body of small mass, simulated by a viscoelastic deformable sphere, has no effect on the motion of the first two and moves in a gravitational field generated by the first two bodies. The scattering of energy when the viscoelastic sphere is deformed leads to the evolution of its orbit and of the angular velocity of motion. In the development of previous results [1] a system of equations is obtained taking into account the second approximation with respect to the small parameter $\mu$, describing the total pattern of the evolution of the motion of the viscoelastic sphere in the restricted circular three-body problem. 

Suppose two point masses $M_1$ and $M_2$, whose masses are equal to unity and $\mu$ ($\mu \ll 1$), move in circular orbits under the action of Newtonian gravitation around a common centre of mass $O$ in the $OXY$ plane. Suppose $OM_2 = b$ and $OM_1 = \mu b$, while the angle $\alpha$ between the $OX$ axis and the radius vector $OM_2$ of the point $M_2$ varies as

$$\alpha(t) = \frac{\omega_3 t}{1+\mu} + \alpha(0), \quad \omega_3 = \sqrt{\frac{f}{b^3}}$$

where $f$ is the gravitational constant (Fig. 1).

We will further assume that the centre of mass $C$ of the sphere (everywhere, unless otherwise stated, we have in mind a viscoelastic deformable uniform sphere) of mass $m$ and density $\rho$ moves in the $OXY$ plane, and $\mathbf{R}$ is the radius vector of the point $C$. The position of points of the sphere is determined by the vector field

$$\boldsymbol{\zeta}(\mathbf{r},t) = \mathbf{R}(t) + O(t)(\mathbf{r} + \mathbf{u}(\mathbf{r},t))$$

$$\mathbf{R}(t) = \frac{1}{m}\int \boldsymbol{\zeta}(\mathbf{r},t)\rho dx, \quad \int \mathbf{u} dx = 0, \quad \int \operatorname{rot} \mathbf{u} dx = 0, \quad dx = dx_1 dx_2 dx_3 \tag{1}$$

and everywhere henceforth, unless otherwise stated, the integration is carried out over the region $V = \{\mathbf{r}: |\mathbf{r}| < r_0\}$ in $E^3$, occupied by the sphere in the natural underformed state.

Conditions (1) uniquely define the radius vector $\mathbf{R}(t)$ of the centre of mass $C$ of the deformed sphere and the system of coordinates $Cx_1x_2x_3$, relative to which the sphere does not rotate in the integral sense [2]. The operator $O(t) = O(\varphi(t))$ determines the transition from the system of coordinates $Cx_1x_2x_3$ to the system of König axes $C\xi_1\xi_2\xi_3$ and has the form

$$O(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

The kinetic energy of the sphere is represented by the functional

$$T = \frac{1}{2}\int \dot{\boldsymbol{\zeta}}^2 \rho dx = \frac{1}{2}\int [O^{-1}\dot{\mathbf{R}} + \boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) + \dot{\mathbf{u}}]^2 \rho dx; \quad \boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3, \quad \mathbf{e}_3 = (0,0,1)$$

where $\boldsymbol{\omega}$ is the angular velocity vector defined by the equality $\boldsymbol{\omega} \times (\cdot) = O^{-1}\dot{O}(\cdot)$.

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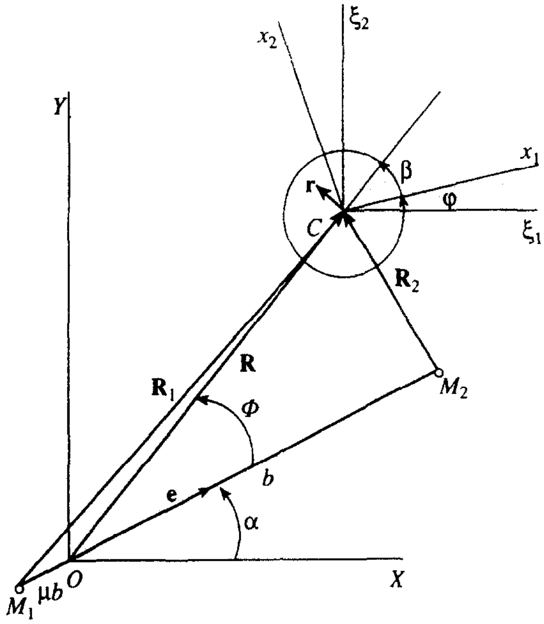


Fig. 1

Taking conditions (1) into account the functional of the kinetic energy can be represented in the form of the sum

$$T = T_2 + T_1 + T_0$$

$$T_2 = \frac{1}{2} m\dot{\mathbf{R}}^2 + \frac{1}{2}\int \dot{\varphi}^2[\mathbf{e}_3 \times (\mathbf{r}+\mathbf{u})]^2 \rho dx, \quad T_1 = \int (\dot{\varphi}\mathbf{e}_3 \times (\mathbf{r}+\mathbf{u}), \dot{\mathbf{u}})\rho dx, \quad T_0 = \frac{1}{2}\int \dot{\mathbf{u}}^2 \rho dx$$

The functional of the potential energy has the form

$$\Pi = -\int \frac{f\rho dx}{\sqrt{(\mathbf{R}_1 + O(\mathbf{r}+\mathbf{u}))^2}} - \int \frac{\mu f\rho dx}{\sqrt{(\mathbf{R}_2 + O(\mathbf{r}+\mathbf{u}))^2}} + \mathscr{E}[\mathbf{u}] \tag{2}$$

where

$$\mathbf{R}_1 = \mathbf{M}_1\mathbf{C} = \mathbf{R} + \mu b\mathbf{e}, \quad \mathbf{R}_2 = \mathbf{M}_2\mathbf{C} = \mathbf{R} - b\mathbf{e}, \quad \mathbf{e} = (\cos\alpha, \sin\alpha, 0)$$

$\mathscr{E}[\mathbf{u}]$ is the functional of the potential energy of elastic deformations, corresponding to the classical theory of elasticity of small deformations,

$$\mathscr{E}[\mathbf{u}] = \int a\left\{\left(\sum_{i=1}^{3} e_{ii}\right)^2 - a_1'\sum_{i<j}^{3} (e_{ii}e_{jj} - e_{ij}^2)\right\}dx, \quad a > 0, \quad 0 < a_1' < 3$$

$$a = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}, \quad a_1' = \frac{2(1-2\nu)}{1-\nu}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$E$ is Young's modulus of elasticity and $\nu$ is Poisson's ratio.

Since $|\mathbf{R}_k| \gg |\mathbf{r} + \mathbf{u}|$ ($k = 1, 2$), the integrands in Eq. (2) can be expanded in series. Confining ourselves to terms of the second order in powers of $|\mathbf{r} + \mathbf{u}|/R_k$ and linear in $|\mathbf{u}|/R_k$, where $R_k = |\mathbf{R}_k|$ ($k = 1, 2$), we obtain

$$\Pi = -\frac{fm}{R_1} - \frac{\mu fm}{R_2} + \frac{f}{R_1^3}\int \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{10},\mathbf{r})(O^{-1}\mathbf{R}_{10},\mathbf{u})\}\rho dx +$$

$$+\frac{\mu f}{R_2^3}\int \{(\mathbf{r},\mathbf{u}) - 3(O^{-1}\mathbf{R}_{20},\mathbf{r})(O^{-1}\mathbf{R}_{20},\mathbf{u})\}\rho dx + \mathscr{E}[\mathbf{u}]$$

$$\mathbf{R}_{k0} = \mathbf{R}_k / R_k, \quad k = 1,2$$

$$R_1 = \sqrt{R^2 + 2\mu Rb\cos\Phi + \mu^2 b^2}, \quad R_2 = \sqrt{R^2 - 2Rb\cos\Phi + b^2}, \quad R = |\mathbf{R}|$$

where $\Phi$ is the angle between the vectors $OC$ and $OM_2$.

We will write the equations of motion in the form Routh equations, using the Poincaré–Andoyer canonical variables, which define the motion of the centre of mass of the sphere and its rotation about the $Cx_3$ axis, and the Lagrangian variables $u_1(\mathbf{r}, t)$, $u_2(\mathbf{r}, t)$, $u_3(\mathbf{r}, t)$, which describe the deformations. Here we will confine ourselves to considering the class of quasi-circular orbits, i.e. orbits with zero eccentricity. It has been shown [1] that such a class of motions exists.

The Andoyer variable $I$, equal to the modulus of the angular momentum vector of the sphere about the point $C$, taken with a plus sign in the case of direct rotation and with a minus sign in the case of inverse rotation, is given by the equation

$$I = \nabla_{\dot\varphi} T = J[\mathbf{u}]\dot\varphi + G^u$$

where

$$J[\mathbf{u}] = \int [\mathbf{e}_3 \times (\mathbf{r} + \mathbf{u})]^2 \rho dx, \quad G^u = \left(\mathbf{e}_3, \int [(\mathbf{r} + \mathbf{u}) \times \dot{\mathbf{u}}] \rho dx\right)$$

The length of the vector $\mathbf{R}$ and the angle it makes with the $OX$ axis, in Poincaré variables $(\Lambda, \lambda)$ for the class of quasicircular orbits is defined by the following relations [1, 3]

$$R = \Lambda^2 (fm^2)^{-1}, \quad \lambda = \Phi + \alpha$$

The Routh functional is defined by the equation

$$\mathcal{R} = T_2 - T_0 + \Pi$$

and is represented in Andoyer–Poincaré variables in the form [1]

$$\mathcal{R} = [I, \Lambda, \varphi, \lambda\, \dot{\mathbf{u}}, \mathbf{u}, \alpha] = \frac{(I - G^u)^2}{2J[\mathbf{u}]} - \frac{f^2 m^3}{2\Lambda^2} - \frac{1}{2}\int \dot{\mathbf{u}}^2 \rho dx + H[\mu, R, \Phi, \beta, \mathbf{u}] + \mathscr{E}[\mathbf{u}]$$

where

$$R = \Lambda^2 (fm^2)^{-1}, \quad \Phi = \lambda - \alpha, \quad \beta = \lambda - \varphi$$

$$H[\mu, R, \Phi, \beta, \mathbf{u}] = F_0 + \int \{F_1(\mathbf{r}, \mathbf{u}) - F_2(\xi_0, \mathbf{r})(\xi_0, \mathbf{u}) - F_3[(\xi_0, \mathbf{r})(\xi_1, \mathbf{u}) + (\xi_1, \mathbf{r})(\xi_0, \mathbf{u})] - $$
$$- F_4(\xi_1, \mathbf{r})(\xi_1, \mathbf{u})\} dx \tag{3}$$

$$\xi_0 = O^{-1}\mathbf{R}_0 = (\cos\beta, \ \sin\beta, \ 0), \quad \xi_1 = O^{-1}\mathbf{e} = (\cos(\beta - \Phi), \ \ \sin(\beta - \Phi), \ 0)$$

$$F_i = F_i(\mu, R, \Phi), \quad i = 0,1,2,3,4$$

The equations of motion of the sphere have the form (everywhere henceforth the subscripts $\varphi$, $\beta$, $\lambda$, etc. denote partial derivatives with respect to the corresponding variables)

$$\dot I = -\mathcal{R}_\varphi = -H_\varphi = H_\beta, \quad \dot\Lambda = -\mathcal{R}_\lambda = -H_\lambda = -H_\Phi - H_\beta$$

$$\dot\Phi = \mathcal{R}_\Lambda - \frac{\omega_3}{1+\mu} = \frac{f^2 m^3}{\Lambda^3} + H_\Lambda - \frac{\omega_3}{1+\mu}, \quad \dot\beta = \mathcal{R}_\Lambda - \mathcal{R}_I = \frac{f^2 m^3}{\Lambda^3} - \frac{I - G^u}{J[\mathbf{u}]} + H_\Lambda \tag{4}$$

$$\int \left[\left(-\frac{d}{dt}\nabla_{\dot{\mathbf{u}}}\mathcal{R} + \nabla_{\mathbf{u}}\mathcal{R} + \nabla_{\dot{\mathbf{u}}} D + \boldsymbol{\lambda}_1\right)\delta\mathbf{u} + \boldsymbol{\lambda}_2 \operatorname{rot}\delta\mathbf{u}\right] = 0, \quad \forall \delta\mathbf{u} \in (W_2^1(V))^3$$

Here $(W_2^1(V))^3$ is the Sobolev space, $D[\dot{\mathbf{u}}] = \chi\mathscr{E}[\dot{\mathbf{u}}]$ is the dissipative functional and $\chi > 0$ is the coefficient of internal viscous friction (the Kelvin–Veight model). The last equation is written in the form of the d'Alembert–Lagrange variational principle and contains two undetermined multipliers $\boldsymbol{\lambda}_1(t)$ and $\boldsymbol{\lambda}_2(t)$, generated by conditions (1).

We will assume that the lowest frequency of natural oscillations of the sphere is much greater than the angular velocities $\dot{\Phi}$ and $\omega_3$. This means that, by an appropriate choice of the scales of the dimensional quantities, the numerical value of the elasticity modulus of the material of the sphere $E$ will be high, while the parameter $\varepsilon = E^{-1}$ is small.

If $\varepsilon = 0$, we have $\mathbf{u}(\mathbf{r}, t) = 0$, and Eqs (4) will look as follows:

$$\dot{l} = 0, \quad \dot{\Lambda} = -F_{0\Phi}, \quad \dot{\beta} = \omega_2 - \omega_1 + F_{0\Lambda}, \quad \dot{\Phi} = \omega_2 - \frac{\omega_3}{1+\mu} + F_{0\Lambda} \tag{5}$$

where

$$\omega_1 = \frac{l}{A}, \quad \omega_2 = \frac{f^2 m^3}{\Lambda^3}, \quad \omega_3 = \sqrt{\frac{f}{b^3}} \tag{6}$$

and $A$ is the moment of inertia of the undeformed sphere about the $Cx_3$ axis. Equations (5) described the motion of the sphere as a rigid body in the classical restricted three-body problem in the class of quasi-circular orbits.

If $\varepsilon \neq 0$, according to the method of separation of motions [2] after the natural oscillations of the sphere have decayed, the solution $\mathbf{u}(\mathbf{r}, t)$ is sought in the form

$$\mathbf{u}(\mathbf{r}, t) = \varepsilon \mathbf{u}_1(\mathbf{r}, t) + \ldots$$

The function $\mathbf{u}_1(\mathbf{r}, t)$, obtained earlier in [1], can be represented in the form of the sum

$$\mathbf{u}_1 = \mathbf{u}_{11} + \mathbf{u}_{12} + \mathbf{u}_{13} \tag{7}$$

where

$$\mathbf{u}_{11} = -\frac{2}{3}\rho\omega_1^2[d_1\mathbf{r}^2 + d_2]\mathbf{r}, \quad \mathbf{u}_{12} = \frac{1}{3}\rho\omega_1^2\mathbf{u}_{120}, \quad \mathbf{u}_{13} \approx \left(1 - \chi\frac{d}{dt}\right)\mathbf{u}_{130}$$

$$\mathbf{u}_{1j0} = a_1(B_{j-1}\mathbf{r}, \mathbf{r})\mathbf{r} + a_2\mathbf{r}^2 B_{j-1}\mathbf{r} + a_3 B_{j-1}\mathbf{r}, \quad j = 2, 3$$

Here

$$d_1 = \frac{(1+\nu)(1-2\nu)}{2(4-3\nu)}, \quad d_2 = -\frac{(3-\nu)(1-2\nu)}{2(4-3\nu)}r_0^2 \tag{8}$$

$$a_1 = \frac{1+\nu}{5\nu+7}, \quad a_2 = -\frac{(1+\nu)(2+\nu)}{5\nu+7}, \quad a_3 = \frac{(1+\nu)(2\nu+3)}{5\nu+7}r_0^2 \tag{9}$$

$$B_1 = \mathrm{diag}\{1, 1, -2\}, \quad B_2 = \| b_{ij} \|_{i,j=1,2,3}$$

$$b_{ii} = \frac{1}{2}[F_1 + (-1)^{i+1}(F_2\cos 2\beta + 2F_3\cos(2\beta - \Phi) + F_4\cos(2\beta - 2\Phi))], \quad i = 1, 2$$

$$b_{12} = b_{21} = \frac{1}{2}[F_2\sin 2\beta + 2F_3\sin(2\beta - \Phi) + F_4\sin(2\beta - 2\Phi)] \tag{10}$$

$$b_{33} = -F_1, \quad b_{13} = b_{31} = b_{23} = b_{32} = 0$$

The matrices of the operators $B_1$ and $B_2$ are symmetrical matrices with zero trace, where

$$B_2\mathbf{r} = -F_1\mathbf{r} + F_2(\boldsymbol{\xi}_0, \mathbf{r})\boldsymbol{\xi}_0 + F_3[(\boldsymbol{\xi}_0, \mathbf{r})\boldsymbol{\xi}_1 + (\boldsymbol{\xi}_1, \mathbf{r})\boldsymbol{\xi}_0] + F_4(\boldsymbol{\xi}_1, \mathbf{r})\boldsymbol{\xi}_1 \tag{11}$$

$$B_2^T = B_2, \quad \mathrm{tr}\, B_2 = -3F_1 + F_2 + 2F_3\cos\Phi + F_4 = 0$$

The function $\mathbf{u}_{13}$ is represented by the first two terms of the power series in $\chi$, assuming that $|\chi\omega_k| \ll 1$ ($k = 1, 2, 3$). Differentiation with respect to time in the expression for the function $\mathbf{u}_{13}$ is carried by virtue of the "unperturbed" system (5).

The solution $\mathbf{u} = \varepsilon\mathbf{u}_1 = \varepsilon(\mathbf{u}_{11} + \mathbf{u}_{12} + \mathbf{u}_{13})$ describes forced oscillations of the sphere. According to the asymptotic method of separation of the motions it is further necessary to substitute this solution into the right-hand sides of Eqs (4), first linearizing them with respect to $\mathbf{u}$ and $\dot{\mathbf{u}}$. Note that the functional H, defined by (3), according to expressions (11) can be represented as follows:

$$H = F_0 - \int (B_2\mathbf{r}, \mathbf{u})dx$$

The system of equations describing the translational-rotational motion of the sphere, taking into account the perturbations connected with the elasticity and dissipation, then takes the form

$$\begin{aligned}
&\dot{I} = -\int (B_{2\beta}\mathbf{r}, \varepsilon\mathbf{u}_1)dx, \quad \dot{\Lambda} = -F_{0\Phi} + \int (B_{2\Phi}\mathbf{r}, \varepsilon\mathbf{u}_1)dx + \int (B_{2\beta}\mathbf{r}, \varepsilon\mathbf{u}_1)dx \\
&\dot{\Phi} = \omega_2 - \frac{\omega_3}{1+\mu} + F_{0\Lambda} - \int (B_{2\Lambda}\mathbf{r}, \varepsilon\mathbf{u}_1)dx \\
&\dot{\beta} = \omega_2 - \omega_1 + \frac{1}{A}\left(\mathbf{e}_3, \int \mathbf{r} \times \varepsilon\dot{\mathbf{u}}_1\rho dx\right) + F_{0\Lambda} - \int (B_{2\Lambda}\mathbf{r}, \varepsilon\mathbf{u}_1)dx + \\
&+2IA^{-2}\int [(\mathbf{r}, \varepsilon\mathbf{u}_1) - (\mathbf{e}_3, \mathbf{r})(\mathbf{e}_3, \varepsilon\mathbf{u}_1)]\rho dx
\end{aligned} \tag{12}$$

The function $\mathbf{u}_1$ is defined by (7).

The next step is to evaluate the triple integrals over the region $V$ on the right-hand sides of Eqs (12) and obtain a closed system of ordinary differential equations in the variables $I$, $\Lambda$, $\Phi$ and $\beta$. We will formulate assertions which enable this procedure to be simplified considerably.

*Lemma 1.* If $B$ is a symmetrical matrix with zero trace, the elements of which are independent of $\mathbf{r}$, and $\mathbf{u} = k(d_1\mathbf{r}^2 + d_2)\mathbf{r}$ ($k$ is a certain multiplier which is independent of $\mathbf{r}$, while the constants $d_1$ and $d_2$ are defined by (8)), then

$$\int (B\mathbf{r}, \mathbf{u})dx = 0$$

*Lemma 2.* If $B$ and $C$ are symmetrical matrices with zero trace, the elements of which are independent of $\mathbf{r}$, and $\mathbf{u} = k[a_1(C\mathbf{r}, \mathbf{r})\mathbf{r} + a_2\mathbf{r}^2C\mathbf{r} + a_3C\mathbf{r}]$ ($k$ is a certain factor independent of $\mathbf{r}$, while the constants $a_1$, $a_2$ and $a_3$ are defined by (9)), then

$$\int (B\mathbf{r}, \mathbf{u})dx = kD_2 \operatorname{tr}[BC], \quad D_2 = \frac{4\pi r_0^7(1+\nu)(9\nu+13)}{105(5\nu+7)}$$

where tr$[BC]$ is the trace of the product of matrices $B$ and $C$.

Lemmas 1 and 2 are proved by direct evaluation of the integrals over the sphere.

System of equations (12), after the above calculations have been carried out, can be represented in the form

$$\begin{aligned}
&\dot{I} = \varepsilon\chi D_2\{T^{\beta\beta}\dot{\beta}_* + T^{\beta\Phi}\dot{\Phi}_* + T^{\beta\Lambda}\dot{\Lambda}_*\} \\
&\dot{\Lambda} = -\dot{I} + \left\{-F_0 + \varepsilon D_2\rho\omega_1^2F_1 + \frac{1}{2}\varepsilon D_2T\right\}_\Phi - \varepsilon\chi D_2\{T^{\Phi\beta}\dot{\beta}_* + T^{\Phi\Phi}\dot{\Phi}_* + T^{\Phi\Lambda}\dot{\Lambda}_*\} \\
&\dot{\Phi} = \omega_2 - \frac{\omega_3}{1+\mu} + \left\{F_0 - \varepsilon D_2\rho\omega_1^2F_1 - \frac{1}{2}\varepsilon D_2T\right\}_\Lambda + \varepsilon\chi D_2\{T^{\Lambda\beta}\dot{\beta}_* + T^{\Lambda\Phi}\dot{\Phi}_* + T^{\Lambda\Lambda}\dot{\Lambda}_*\}
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\dot{\beta} = \omega_2 - \omega_1 + \left\{F_0 - \varepsilon D_2\rho\omega_1^2F_1 - \frac{1}{2}\varepsilon D_2T\right\}_\Lambda + \varepsilon\chi D_2\{T^{\Lambda\beta}\dot{\beta}_* + T^{\Lambda\Phi}\dot{\Phi}_* + T^{\Lambda\Lambda}\dot{\Lambda}_*\} + \\
&+2\varepsilon\rho\frac{I}{A^2}\left\{\frac{2}{3}\rho\omega_1^2(2D_1 + D_2) + D_2[F_1 - \chi F_{1\Phi}\dot{\Phi}_* - \chi F_{1\Lambda}\dot{\Lambda}_*]\right\}
\end{aligned} \tag{14}$$

Here

$$\dot\beta_* = \omega_2 - \omega_1 + F_{0\Lambda}, \quad \dot\Phi_* = \omega_2 - \frac{\omega_3}{1+\mu} + F_{0\Lambda}, \quad \dot\Lambda_* = -F_{0\Phi}, \quad D_1 = \frac{8\pi r_0^7(1-2\nu)}{105}$$

$$T = \mathrm{tr}[B_2^2], \quad T^{\beta\beta} = \mathrm{tr}[B_{2\beta}B_{2\beta}], \quad T^{\beta\Phi} = \mathrm{tr}[B_{2\beta}B_{2\Phi}] \ \text{etc.}$$

Using formulae (10), we obtain

$$T = 2[3F_1^2 + (F_3^2 - F_2F_4)\sin^2\Phi]$$

$$T^{\beta\beta} = 2[9F_1^2 + 4(F_3^2 - F_2F_4)\sin^2\Phi]$$

$$T^{\beta\Phi} = -[2(F_2F_{3\Phi} - F_3F_{2\Phi} + F_3F_{4\Phi} - F_4F_{3\Phi})\sin\Phi + 2F_4^2 +$$
$$+(F_2F_{4\Phi} - F_4F_{2\Phi})\sin 2\Phi + 2F_2(F_3\cos\Phi + F_4\cos 2\Phi) + 4F_3^2 + 6F_3F_4\cos\Phi]$$

$$T^{\beta\Lambda} = -[2(F_2F_{3\Lambda} - F_3F_{2\Lambda} + F_3F_{4\Lambda} - F_4F_{3\Lambda})\sin\Phi + (F_2F_{4\Lambda} - F_4F_{2\Lambda})\sin 2\Phi]$$

$$T^{\Phi\Phi} = \frac{3}{2}(F_{1\Phi})^2 + \frac{1}{2}(F_{2\Phi})^2 + 2(F_{3\Phi})^2 + \frac{1}{2}(F_{4\Phi})^2 + 2F_3^2 + 2F_4^2 +$$
$$+2F_{2\Phi}(F_3\cos\Phi)_\Phi + F_{2\Phi}(F_4\cos 2\Phi)_\Phi + 2F_{4\Phi}(F_{3\Phi}\cos\Phi + F_3\sin\Phi) +$$
$$+4F_4(F_3\cos\Phi - F_{3\Phi}\sin\Phi)$$

$$T^{\Phi\Lambda} = \frac{3}{2}F_{1\Phi}F_{1\Lambda} + \frac{1}{2}F_{2\Phi}F_{2\Lambda} + 2F_{3\Phi}F_{3\Lambda} + \frac{1}{2}F_{4\Phi}F_{4\Lambda} + (F_{2\Phi}F_{3\Lambda} + F_{2\Lambda}F_{3\Phi})\cos\Phi +$$
$$+\frac{1}{2}(F_{2\Phi}F_{4\Lambda} + F_{4\Phi}F_{2\Lambda})\cos 2\Phi + F_3\sin\Phi(F_{4\Lambda} - F_{2\Lambda}) + (F_{3\Phi}F_{4\Lambda} + F_{3\Lambda}F_{4\Phi})\cos\Phi -$$
$$-2F_4\sin\Phi(F_{2\Lambda}\cos\Phi + F_{3\Lambda})$$

$$T^{\Lambda\Lambda} = \frac{3}{2}(F_{1\Lambda})^2 + \frac{1}{2}(F_{2\Lambda})^2 + 2(F_{3\Lambda})^2 + \frac{1}{2}(F_{4\Lambda})^2 +$$
$$+2(F_{2\Lambda}F_{3\Lambda} + F_{3\Lambda}F_{4\Lambda})\cos\Phi + F_{2\Lambda}F_{4\Lambda}\cos 2\Phi$$

The functions $F_i (i = 0, 1, \ldots, 4)$ depend on the variables $\Lambda$ and $\Phi$ and are independent of the variable $\beta$. Hence, the right-hand sides of Eqs (13) and (14) are also independent of the variable $\beta$. System of equations (13) is a closed system of ordinary differential equations in the variables $I$, $\Lambda$ and $\Phi$, containing the small parameters $\varepsilon$ and $\mu$, while Eq. (14) is separated from system (13) and can be integrated after determining the functions $I$, $\Lambda$ and $\Phi$ (as functions of time) from system (13).

We will further consider system of equations (13) as a system with a small parameter $\mu$ for a fixed value of the parameter $\varepsilon$.

When $\mu = 0$ we obtain

$$\dot I = 18\varepsilon\chi D_2\rho^2\omega_2^4(\omega_2 - \omega_1), \quad \dot\Lambda = -18\varepsilon\chi D_2\rho^2\omega_2^4(\omega_2 - \omega_1) \tag{15}$$

$$\dot\Phi = \omega_2 - \omega_3 + 6\varepsilon D_2\rho^2\Lambda^{-1}\omega_2^2(\omega_1^2 + 6\omega_2^2) \tag{16}$$

The quantities $\omega_1$, $\omega_2$ and $\omega_3$ are defined by formulae (6). Note that the right-hand sides of Eqs (15) and (16) are independent of the angular variables.

The solution of system of equations (13) will be sought in the form [4]

$$I = J_1 + \mu N_{11} + \mu^2 N_{12} + \ldots, \quad \Lambda = J_2 + \mu N_{21} + \mu^2 N_{22} + \ldots$$

$$\Phi = \psi + \mu M_1 + \mu^2 M_2 + \ldots \tag{17}$$

where the function $N_{ik} = N_{ik}(J, \psi)$, $M_k = M_k(J, \psi)$ $(i = 1, 2, k = 1, 2, \ldots)$ are $2\pi$-periodic in the variable $\psi$ and have zero mean with respect to this variable, and $J = (J_1, J_2)$.

As a function of time, $J_1, J_2, \psi$ are given by the differential equations

$$\begin{aligned} \dot{J}_1 &= A_{10}(J)+\mu A_{11}(J)+\mu^2 A_{12}(J)+\ldots \\ \dot{J}_2 &= A_{20}(J)+\mu A_{21}(J)+\mu^2 A_{22}(J)+\ldots \\ \dot{\psi} &= \omega(J)+\mu B_1(J)+\mu^2 B_2(J)+\ldots \end{aligned} \tag{18}$$

We will expand the functions $F_i (i = 0, 1, \ldots, 4)$ in series in powers of the small parameter $\mu$. Confining ourselves to terms of the second order of smallness in $\mu$, we obtained

$$F_0 = \mu f_{01} + \mu^2 f_{02}, \quad F_1 = f_{10} + \mu f_{11} + \mu^2 f_{12}, \quad F_2 = f_{20} + \mu f_{21} + \mu^2 f_{22}$$
$$F_3 = \mu f_{31} + \mu^2 f_{32}, \quad F_4 = \mu f_{41} + \mu^2 f_{42}$$

where

$$\begin{aligned} &f_{01} = \Lambda\omega_2(q\cos\Phi - p), \quad f_{02} = \frac{\Lambda\omega_2}{2}q^2(1-3\cos^2\Phi) \\ &f_{10} = \rho\omega_2^2, \quad f_{11} = \rho\omega_2^2(p^3 - 3q\cos\Phi), \quad f_{12} = \frac{3\rho\omega_2^2}{2}q^2(5\cos^2\Phi - 1) \\ &f_{20} = 3\rho\omega_2^2, \quad f_{21} = 3\rho\omega_2^2(p^5 - 5q\cos\Phi), \quad f_{22} = \frac{15\rho\omega_2^2}{2}q^2(7\cos^2\Phi - 1) \\ &f_{31} = 3\rho\omega_2^2 q(1-p^5), \quad f_{32} = -15\rho\omega_2^2 q^2\cos\Phi \\ &f_{41} = 3\rho\omega_2^2 q^2 p^5, \quad f_{42} = 3\rho\omega_2^2 q^2 \\ &q = \frac{b}{R} = \left(\frac{\omega_2}{\omega_3}\right)^{2/3} = \frac{fm^2 b}{\Lambda^2}, \quad p = \frac{1}{\sqrt{1-2q\cos\Phi+q^2}} \end{aligned} \tag{19}$$

Substituting expansions (17) and (18) into Eq. (13) and equating coefficients of like powers of the small parameter $\mu$, we obtain a system of equations which determines the unknown functions $A_{1k}, A_{2k}, B_k, N_{1k}, N_{2k}, M_k$ $(k = 1, 2, \ldots)$. The functions $A_{10}(J), A_{20}(J)$ and $\omega(J)$ of the zeroth approximation in the small parameter $\mu$ have the form

$$A_{10}(J) = -A_{20}(J) = 18\varepsilon\chi D_2\rho^2\omega_2^4(\omega_2 - \omega_1)$$
$$\omega(J) = \omega_2 - \omega_3 + 6\varepsilon D_2\rho^2\Lambda^{-1}\omega_2^2(\omega_1^2 + 6\omega_2^2)$$

where the variables $I$ and $\Lambda$ are replaced by the variables $J_1$ and $J_2$ respectively in the expressions for $\omega_1$ and $\omega_2$.

The system of differential equations of the zeroth approximation with respect to the small parameter $\mu$ in the variables $J_1$ and $J_2$ has the form

$$\dot{J}_1 = A_{10}(J), \quad \dot{J}_2 = A_{20}(J) \tag{20}$$

and describes the motion of a sphere in a central Newtonian force field in the class of quasi-circular orbits (the class of motions when the rotation of the sphere occurs about the normal to the plane of the orbit is considered). System of equations (20) has an asymptotically stable stationary solution, corresponding to gravitational stabilization of this sphere in a circular orbit with centre at the point $O$ (which coincides with the point $M_1$ in the zeroth approximation with respect to $\mu$). The radius $R_0$ of this orbit is given by the system of equations

$$J_1 + J_2 = G_0, \quad J_1 / A = f^2 m^3 / J_2^3 \tag{21}$$

Here $G_0$ is the angular momentum of the sphere about the point $O$, which remains unchanged in the zeroth approximation with respect to the small parameter $\mu$. When the inequality

$$G_0 > \tfrac{4}{3}(3Af^2m^3)^{1/4}$$

is satisfied, system of equations (21) has two solutions $(J_{11}, J_{21})$ and $(J_{12}, J_{22})$ $(J_{21} < J_{22})$. The asymptotically stable stationary solution of system of equations (20) corresponds to the larger value $J_{22}$. Hence

$$R_0 = f^2m^3 / J_{22}^3$$

In the $\mu$-neighbourhood of this stationary solution, it is necessary to take into account the following approximation with respect to $\mu$ in expansions (18). Assuming

$$A_{10}(J) = \varepsilon\chi\mu\tilde{A}_0(J), \quad A_{20}(J) = -\varepsilon\chi\mu\tilde{A}_0(J)$$

we will write the system of equations which determine the unknown functions $A_{11}$, $A_{21}$, $N_{11}$, $N_{21}$ and $M_1$ of the first approximation with respect to the small parameter $\mu$

$$\begin{aligned}
&A_{11} + N_{11\psi}\omega = \varepsilon\chi D_2\{18f_{10}^2 f_{01,2} - (2f_{20}f_{31}\sin\psi + f_{20}f_{41}\sin 2\psi)_\psi(\omega_2 - \omega_3)\}\\
&A_{21} + N_{21\psi}\omega = \{-f_{01} + \varepsilon D_2 Q_1\}_\psi -\\
&-\varepsilon\chi D_2\{18f_{10}^2 f_{01,2} - (2f_{20}f_{31}\sin\psi + f_{20}f_{41}\sin 2\psi)_\psi(\omega_2 - \omega_3)\}\\
&B_1 + M_{1\psi}\omega = \omega_{,1}N_{11} + \omega_{,2}N_{21} + \omega_3 + \{f_{01} - \varepsilon D_2 Q_1\}_{,2} +\\
&+\varepsilon\chi D_2\{f_{20,2}(2f_{11} - f_{41}\sin^2\psi)(\omega_2 - \omega_3) - 6f_{01}(f_{10,2})^2\}_\psi
\end{aligned} \tag{22}$$

where

$$Q_1 = \rho\omega_1^2 f_{11} + 3f_{10}(2f_{11} - f_{41}\sin^2\psi)$$

while in the functions $f_{jk}$ the variables $\Lambda$ and $\Phi$ are replaced by the variables $J_2$ and $\psi$ respectively. The subscript $i$ $(i = 1, 2)$ after the comma denotes a partial derivative with respect to $J_i$: $f_{01,2} = \partial f_{01}/\partial J_2$, etc.

The functions $A_{11}$ and $A_{21}$ of the first approximation with respect to the small parameter $\mu$, which define the evolution of the action variables, have the form

$$A_{i1} = 18(-1)^{i+1}\varepsilon\chi D_2 f_{10}^2\langle f_{01,2}\rangle, \quad i = 1, 2$$

where

$$\langle\cdot\rangle = \frac{1}{2\pi}\int_0^{2\pi}(\cdot)d\psi, \quad \langle f_{01,2}\rangle = 2\omega_2 a(q), \quad a(q) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - q\cos\psi}{(1 - 2q\cos\psi + q^2)^{3/2}}d\psi$$

When $q = 1$ the integral defining the function $a(q)$ diverges. When $q < 1$, i.e. for outer orbits of the sphere, the function $a(q)$ takes positive values, and when $q > 1$, i.e. for inner orbits of the sphere, the function $a(q)$ takes negative values.

Approximate equations describing the evolution of the variables $I$ and $\Lambda$, taking into account terms of the first approximation with respect to the small parameter $\mu$, have the form

$$\dot{J}_i = 18(-1)^{i+1}\varepsilon\chi D_2\rho^2\omega_2^4\{(\omega_2 - \omega_1) + 2\mu\omega_2 a(q)\}, \quad i = 1, 2 \tag{23}$$

It follows from system (23) that $\dot{J}_1 + \dot{J}_2 = 0$, i.e. the angular momentum $G_0$ about the point $O$ is also conserved in the first approximation with respect to the small parameter $\mu$. System of equations (23), like system (20), possesses an asymptotically stable solution, corresponding to the motion of the sphere for which its centre of mass moves in an orbit of radius $R_1$, which differs from $R_0$, with angular velocity

$$\omega_1 = \omega_2(1 + 2\mu a(q)) \tag{24}$$

It follows from the properties of the function $a(q)$ and from the fact that in Eqs (23) the angular momentum $G_0$ is conserved, that the stationary orbit of radius $R_1$ is shifted only slightly with respect to the orbit of a body of mass $\mu$ compared with the orbit of radius $R_0$, obtained in the zeroth approximation with respect to the small parameter $\mu$. Hence, at the first stage of the slow dissipative evolution, which occurs at a rate of $\varepsilon\chi\mu$, the radius of the quasi-circular orbit of the sphere, which is inside the orbit of the body of mass $\mu$, increases, while the radius of the quasi-circular orbit of the sphere, situated outside the orbit of the body of mass $\mu$, decreases, tending to its new stationary value $R_1$.

When the sphere moves in an orbit of radius $R_1$ with an angular velocity given by expression (24), a further loss of energy occurs. In the $\mu^2$-neighbourhood of this attractor the following approximation with respect to $\mu$ must be taken into account in expansions (18).

Assuming

$$A_{i0}(J)+\mu A_{i1}(J)=(-1)^{i+1}\varepsilon\chi\mu^2\tilde{A}_i(J),\quad i=1,2$$

we obtain the sum of the functions $A_{12}(J)$ and $A_{21}(J)$ of the second approximation with respect to the small parameter $\mu$; this sum defines the evolution of the kinetic moment $G_0$. The corresponding equation of the second approximation has the form

$$\begin{aligned}&A_{12}+A_{22}+(N_{11.1}+N_{21.1})A_{11}+(N_{11.2}+N_{21.2})A_{21}+(N_{11\psi}+N_{21\psi})B_1+(N_{12\psi}+N_{22\psi})\omega=\\&=-(f_{01}-\varepsilon D_2Q_1)_{\psi\psi}M_1-(f_{01}-\varepsilon D_2Q_1)_{\psi,2}N_{21}+(\varepsilon D_2Q_1)_{\psi,1}N_{11}-\\&-\varepsilon\chi D_2\langle f_{01,2}\rangle\{2f_{20}f_{31}\sin\psi+f_{20}f_{41}\sin 2\psi\}_\psi+(-f_{02}+\varepsilon D_2Q_2)_\psi+\\&+\varepsilon\chi D_2\{(2f_{20}f_{31}\sin\psi+f_{20}f_{41}\sin 2\psi)_\psi f_{01,2}+f_{20,2}f_{01\psi}[2f_{11}-f_{41}\sin^2\psi]_\psi\}-\\&-\varepsilon\chi D_2\left\{\frac{3}{2}(f_{11\psi})^2+\frac{1}{2}(f_{21\psi})^2+2(f_{31\psi})^2+\frac{1}{2}(f_{41\psi})^2+2f_{31}^2+2f_{41}^2+f_{21\psi}(2(f_{31}\cos\psi)_\psi+\right.\\&\left.+(f_{41}\cos 2\psi)_\psi)+2f_{41\psi}(f_{31\psi}\cos\psi+f_{31}\sin\psi)+4f_{41}(f_{31}\cos\psi-f_{31\psi}\sin\psi)\right\}(\omega_2-\omega_3)\end{aligned}\tag{25}$$

where

$$Q_2=\rho\omega_1^2f_{12}+3f_{11}^2+6f_{10}f_{12}+(f_{31}^2-f_{20}f_{42}-f_{21}f_{41})\sin^2\psi$$

while in the functions $f_{jk}$ the variables $\Lambda$ and $\Phi$ are replaced by the variables $J_2$ and $\psi$ respectively.

Since the average of the functions $N_{ij}$ is equal to zero, to determine the sum of the functions $A_{12}(J)$ and $A_{22}(J)$ it is necessary to average the right-hand side of (25) with respect to the variable $\psi$.

Terms on the right-hand side of (25), containing coefficients of the function of the first approximation $N_{11}$, $N_{21}$ and $M_1$ can be converted as follows:

$$\langle U_\psi V\rangle=-\langle UN_\psi\rangle$$

Hence, expressing the derivatives $N_{11\psi}$, $N_{21\psi}$, $M_{1\psi}$ of the system of equations of the first approximation (22) and taking equalities (19) into account, we obtain

$$A_{12}+A_{22}=18\varepsilon\chi D_2\rho^2\omega_2^5b(q)$$

where

$$b(q)=\langle h_1(q,\psi)+h_2(q,\psi)+h_3(q,\psi)-h_4(q,\psi)-h_5(q,\psi)\rangle$$

$$h_1(q,\psi)=\frac{2q^{3/2}}{q^{3/2}-1}\left\{2a_1(q,\psi)+\frac{3q^{3/2}}{q^{3/2}-1}(q\cos\psi-p)\right\}\{a(q)-a_1(q,\psi)\}$$

$$a_1(q,\psi)=p^3(1-q\cos\psi)-2q\cos\psi$$

$$\begin{aligned}h_2(q,\psi)=&\left\{4a_1(q,\psi)+\frac{3q^{3/2}}{q^{3/2}-1}(q\cos\psi-p)\right\}\{5q^2p^7(1-q\cos\psi)\sin^2\psi+\\&+q(1-p^5)\cos\psi+q^2p^5\cos 2\psi\}\end{aligned}$$

$$h_3(q,\psi)=6q^2(1-p^3)\{2(1-p^5-qp^5\cos\psi)+5q^2p^7\sin^2\psi\}\sin^2\psi$$

$$h_4(q,\psi)=\frac{12q^{3/2}}{q^{3/2}-1}q^2(1-p^3)^2\sin^2\psi$$

$$h_5(q,\psi)=(1-q^{-3/2})\sum_{k=1}^{5} g_k(q,\psi)$$

$$g_1(q,\psi)=\frac{1}{4}q^2\{3(1-p^5)^2+25(1-p^7)^2+100q^2p^{14}+25q^4p^{14}\}\sin^2\psi$$

$$g_2(q,\psi)=q^2(1-p^5)^2+q^4p^{10}$$

$$g_3(q,\psi)=5q^2(1-p^7)\{qp^5(5p^2-2)\cos\psi+p^5-1-\tfrac{5}{2}q^2p^7\cos 2\psi\}\sin^2\psi$$

$$g_4(q,\psi)=-5q^4p^7\{5qp^7\cos\psi+1-p^5\}\sin^2\psi$$

$$g_5(q,\psi)=2q^3p^5\{(1-p^5)\cos\psi-5qp^7\sin^2\psi\}$$

Graphs of the function $b(q)$ for $q<1$ and $q>1$ are shown in Fig. 2. For $q<1$ (for outer orbits of the sphere) the function $b(q)$ is positive, and for $q>1$ (for inner orbits of the sphere) it is negative.

At the second stage of the slow evolution the following relations hold

$$J_1=Af^2m^3J_2^{-3}+O(\mu),\quad J_2^2=fm^2bq^{-1}+O(\mu)$$

from which we obtain, apart from small terms of the order of $\mu$, a relation between the average value of the kinetic moment $G_0=J_1+J_2$ and $q$ in the form

$$G_0(q)=m(fb/q)^{1/2}+Af^{1/2}(b/q)^{-3/2},\quad q<q_*=bm^{1/2}(3A)^{-1/2} \tag{26}$$

The value of $q$ must be chosen to be less than $q^*$ ($q^*$ is the point where the function $G_0(q)$ is a minimum), which corresponds to a stable circular orbit of large radius in the unperturbed problem when $\mu=0$. Relation (26), together with the equation of the second approximation

$$\dot{G}_0=18\varepsilon\chi\mu^2D_2\rho^2\omega_2^5(q)b(q) \tag{27}$$

determines the evolution of the average angular momentum of the sphere. The function $G_0(q)$ and the inverse function $q(G_0)$ when $q<q^*$ are positive and decrease monotonically. It follows from Eq. (27), taking (26) and the properties of the function $b(q)$ into account, that the average angular momentum $G_0$ increases for the outer orbits of the sphere and decreases for the inner orbits. We can conclude from the form of the relation $q(G_0)$ that the radius increases monotonically for the outer orbits and decreases monotonically for the inner orbits.

The results obtained using the approximate equations hold for ranges of variation of the radius of the quasi-circular orbits, with the exception of the neighbourhood of the point $b$, in which libration points occur, since in this case the procedure of averaging over the fast angular variable $\Phi$ and expansion in series of one of the gravitational potentials turns out to be incorrect.

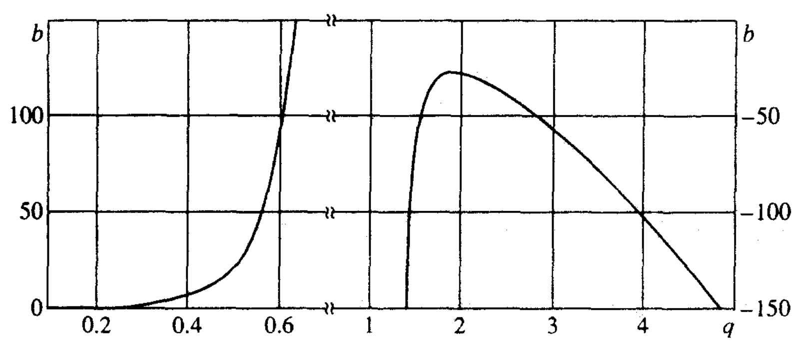


Fig. 2

In conclusion, we note that the nature of the evolution of the quasi-circular orbits of a viscoelastic sphere obtained above agrees with the theorem of the variation of the generalized energy in a uniformly rotating system of coordinates in which the attracting centres are fixed.

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